Chapter 24 THE TERM STRUCTURE OF INTEREST RATES

Here comes a curveball.

Conventional financial theory, portfolio theory and the CAPM, which were presented in Chapter 21, are concerned with the notion of interest rates and reducing it to the level of a factor that is exogenous to their models, namely the risk-free rate. But the risk-free rate is by no means a given variable, and there is no financial instrument in existence which allows investors to completely escape risk.

Moreover, because it is a single-period model, the CAPM draws no distinction between short-term and long-term interest rates. As has been discussed, a money-market fund does not offer the same annual rate of return as a 10-year bond. An entire body of financial research is devoted to understanding movements in interest rates and, in particular, how different maturities are linked. This is the study of how the yield curve is formed.

> Section 24.1 FIXED-INCOME SECURITIES AND RISK

Investing in debt securities is not risk free, although it is much less risky than options or even stocks. There are at least three risks involved in debt securities:

- inflation risk;
- the risk of a change in interest rates if the security's maturity is different from the investment horizon; and
- counterparty (or default) risk.

Counterparty risk is ignored in yield curves, because generally they are based on government debt, and therefore the risk is considered to be negligible. The two other risks play a more important part in interest rate structure.

1/ FIXED-INCOME INSTRUMENTS AND INFLATION

When an investor buys an Italian government zero-coupon bond and keeps it until maturity, he has sure and advance knowledge of the sum he will receive when the bond is redeemed (assuming that the state does not go bankrupt). However, he is not certain of that sum's future purchasing power.

Let us take the example of a zero-coupon bond redeemable for $\in 1000$ in January 2029. If you have paid $\in 377$ for this bond in January 2009, your return will be 5% per year if you keep the bond until maturity. In 2009, the market was pricing in long-term inflation at about 2.5%. This means that the price of an asset worth $\in 1$ in 2009 (equivalent to the value of a subway ticket) should be worth $\in 1.64$ in 2029.¹ So you invested $\in 377$ today in your zero-coupon bond believing that you will be able to buy 610 subway tickets (1000/1.64) in 2029.

Unfortunately, if between 2009 and 2029 inflation is not 2.5% per year, but 4% per year, you will be able to buy just 456 tickets,² not 610. By investing in a long-term bond, you have frozen your nominal interest rate (5%), but not the interest rate after inflation that you will ultimately receive. This rate is called real interest rate.

The relationship between nominal interest rate and real interest rate can be expressed as follows:

Real interest rate = $\frac{1 + \text{Nominal interest rate}}{1 + \text{Inflation rate}} - 1$

If inflation is not too high, the equation can be simplified as follows:

Real interest rate = Nominal interest rate - Inflation rate

2/ FIXED-INCOME SECURITIES AND THE TIMEFRAME FOR INVESTMENT

An investor who does not want to get trapped by an unexpected upturn in inflation can invest his €377 for one year, and then repeat the transaction every year. He will thus be sure that the rate at which he invests will reflect anticipations of inflation. Hence, if the one-year nominal rate is 4% (composed of a real rate of 2% and inflation of 2%) and if inflation rises to 2.5%, short-term rates will probably move to 4.5%. Repeated investment in short-term, fixed-income securities limits the risk of an unexpected upturn in inflation. Hence, our investor, with his €377 invested at 4% and with inflation of 2%, can buy 558 subway tickets in 20 years. This is true even if inflation rises to 2.5%, as the short-term rate would then be 4.5%, with the real long-term unchanged at 2%.

However, this strategy exposes the investor to trends in real short-term interest rates. It is possible that, with constant 1% inflation, real interest rates will move to 1%. The investor's nominal rate would then be 2% and in 20 years, even without any change in inflation, he would only be able to buy 459 subway tickets and not the 558 he was expecting.

Investors also choose their investment timeframe on the basis of liquidity preferences. Repeated short-term investments secure a certain measure of liquidity in exchange for uncertainty about the ultimate rate of return. To secure a guaranteed return, an investor in long-term bonds must keep her bonds until maturity.

3/ CHOOSING RISK

Thus, it can be seen that investors must choose their type of risk. They can elect to have the maturity of their investment coincide with their investment timeframe, and thus expose themselves to inflation risk (but not interest rate risk), or they can choose a short-term timeframe and renew their investment on a regular basis, and thus expose themselves to interest rates (but not inflation).

 $\begin{array}{l} {\bf 1} \ 1.64 = l \times \\ (\,l + 2.5\%)^{20} \end{array}$

2 $456 = 1000/(1+4\%)^{20}$

Depending upon the timeframe chosen, an investor will not consider short-term and long terms rates in the same way:

- The long-term investor is willing to receive a lower yield to maturity than a shortterm investor who will have to renew constantly, considering that "normally" longterm rates are below short-term rates.
- The short-term investor, on the other hand, will consider it "normal" that short-term rates should be below long-term rates.

Only a bond with a redemption value and interest rate indexed to inflation can protect against unexpected changes in inflation and ensure an a fixed annual rate, as long as the issuer does not go bankrupt and the bond's maturity is the same as the investment timeframe.

Section 24.2 THE DIFFERENT INTEREST RATES CURVES

1/HOW A BOND BREAKS DOWN

Consider a bond issued by Lafarge on 28 May 2008 paying 6.125% annual interest on a €50,000 face value and a 7-year maturity.

Like any financial security, the Lafarge bond is a cash flow timetable. But in another light, it can also be thought of as a **portfolio of zero-coupon bonds**.³ This "portfolio" is based on a zero-coupon bond that matures 28 May 2009 with a redemption value of €3065.5, a zero-coupon bond that matures 28 May 2010 with a redemption value of €3065.5, and so on, concluding with a zero-coupon bond that matures 28 May 2015 with redemption value of €53,065.5.

Each of these zero-coupon bonds can be valued individually. The sum of the values is equal to the value of the Scania bond (otherwise arbitrage traders would quickly reestablish the equilibrium). The present value of our Scania bond can thus be calculated as follows:

$$PV = \sum_{t=1}^{5} \frac{F_t}{(1+r_t)^t}$$

where F_t is the cash flows of year $i \in 3065.5$ the first 6 years and $\notin 53,065.5$ the 7th year) and r_t is the market rate for zero-coupon bonds with a maturity of N years. The single rates of interest on zero-coupon bonds are named **spot rates**. The series of spot rates r_1 , r_2 etc. is *one* way of expressing the **term structure** of interest rates.

Rather than discounting each of the payments at a different rate of interest, it could be possible to find a single rate of discount that would produce the same result. This rate is defined as **yield to maturity**, and it is in fact exactly the same as the internal rate of return but masqueraded under another name!

However, the reader should be aware that it is not entirely correct to use the yield to maturity to determine bond prices. Why?

1 Because the bondholder may demand different rates of return for different periods. Unless the two bonds offer exactly the same set of cash flows, they will probably 3 That is, the bond has only one final stream with both capital and capitalised interest. have different yields to maturity. The yield to maturity thus represents only a rough guide to the appropriate yield on another bond.

2 Because the yield to maturity does not determine bond prices. Actually, it is the other way round. The value of any package of cash flows is determined by discounting each cash flow at the appropriate spot rate. Then, *given the value*, we could compute the yield to maturity that, like all averages, summarises the relevant information contained in the term structure of interest rates.

Forward rates are implicit in the spot rate curve. In general, if we are given spot rates, r_1 and r_2 for years 1 and 2, we can determine the forward rate f_2 , such that:

$$(1+r_2)^2 = (1+r_1) \times (1+f_2)$$

We solve for f_2 , yielding:

$$f_2 = \frac{(1+r_2)^2}{(1+r_1)} - 1$$

As previously discussed, one important characteristic of financial securities is maturity. Clearly, an investor will not demand the same return on a 7-year investment as he would on a 1-year one, because he does not expose himself to the same risk. What return should he demand for the short, medium and long term and, more generally, for each possible maturity?

2/ CURVE OF ZERO-COUPON RATES AND SWAP CURVE

By charting the interest rate for the same categories of risk at all maturities, the investor obtains the yield curve that reflects anticipations of all financial market operators.

However, if bond market data were used to calculate yields to maturity, then the coupons would introduce a bias. The yield curve shows the interest rate on instruments with different maturities and different coupons. Using a zero-coupon curve allows one to track overall changes in yields to maturity, while offering the advantage of describing more precisely the changes caused by market anticipation.

Zero-coupon bonds for each maturity have recently developed very quickly, although they are rarely listed and, when they are, they are too illiquid for their yield to maturity to be significant. So a zero-coupon yield curve is only based on listed and liquid bonds (such as government bonds). One common method is commonly obtained with the **bootstrapping model**, which first studies bonds maturing in 1 year. Their yield to maturity is necessarily a zero-coupon yield (as there is only one cash flow, one year out). Then by observing the price of a 2-year bond, we can figure the yield to maturity of a 2-year zero-coupon bond. The entire curve is obtained by repeating the calculation for each maturity.⁴

With the development of inflation-indexed government bonds, it should soon be possible to estimate the real yield curve on zero-coupon bonds. Inflation-indexed bonds are a financial innovation that disassociates two risks that had been intertwined: the coupon reinvestment risk and the inflation risk.

Interest rate swaps⁵ allow banks to trade fixed-rate-based interest streams for variable-rate ones among themselves or with clients. This development has led to a much larger market than the one for government bonds, as many governments are seeking to

4 That is, the bond has only one final stream with both capital and capitalised interest.

5 See Chapter 47. reduce their debt. More and more often, yield curves are calculated on the basis of interest rate swaps, and these are becoming the benchmark.

3/THE VARIOUS YIELD CURVES

The concept of premium helps explain why the interest rate of any financial asset is generally proportional to its maturity.



SOME YIELD CURVES AS OF JUNE 2008

Generally speaking, the yield curve reflects the market's anticipation about:

- long-term inflation;
- the central bank's monetary policy; and
- the country's issuing debt management policy.

Hence, during a period of economic recovery, the yield curve tends to be "normal" (i.e. long yields are higher than short yields). The steepness of the slope depends on:

- how strong an expected recovery is;
- what expectations the market has about the risk of inflation; and
- the extent to which the market expects a rapid tightening in central banks' intervention rates (to calm inflationary risks).

The dollar curve's upward slope in 2008 is due to the extremely low levels reached by short-term rates in US, following central bank interventions to avoid a major economic downturn after the subprime crisis broke out in summer 2007.

In contrast, when a recession follows a period of growth, the yield curve tends to reverse itself (with long-term rates falling below short-term rates). The steepness of the negative slope depends on:

• how strong expectations of recovery are;

- how credible the central bank's policy is (i.e. how firm the central banks are in fighting inflation); and
- the extent to which inflationary trends appear to be diminishing (despite the recession, if inflationary trends are very strong then long-term rates will tend to remain stable, and the curve could actually be flat for a some time).

The almost flat euro curve in 2008 is due to inflation anticipations and fear of an economic slowdown: long-term rates remain steady and so the curve is flat.

Lastly, when rates are low, the curve cannot remain flat for any length of time because investors will buy fixed-rate bonds. As long as investors expect that their capital gain, which is tied to falling long-term rates, is more than the cost of short-term financing, then they will continue to purchase the fixed-rate bonds. However, when long-term rates seem to have reached a lower limit, these expectations will disappear because investors will demand a differential between long-term and short-term rates yield on their investment. This results in:

- either a rebound in long-term rates; or
- stable long-term rates if short-term rates fall because of central bank policies; and
- a steepening in the curve, the degree of which will depend on the currency.

The shape of the yield curve can also depend upon anticipation of political events. Hence, the downward slope in the Polish yield curve is clearly due to market anticipation of Poland's entry into the Euro area, leading to a convergence in long-term rates.

Section 24.3 RELATIONSHIP BETWEEN INTEREST RATES AND MATURITIES

1/ Relation between interest rate and maturities

By no means are short-term and long-term rates completely disconnected. In fact, there is a fundamental and direct link between them.

About 20 years ago, this relationship was less apparent and common consensus favoured the **theory of segmentation**, which said that supply and demand balanced out across markets, with no connection among them, e.g. the long-term bond market and the short-term bond market.

As seen above, this theory is generally no longer valid, even though each investor will tend to focus on his own timeframe. It is worthwhile reviewing the basic mechanisms. For example, an investor who wishes to invest on a 2-year time basis has two options:

- he invests for 2 years at today's fixed rate, which is the interest rate for any 2-year investment; or
- he invests the funds for 1 year, is paid the 1-year interest rate at the end of the year, and then repeats the operation.

In a **risk-free environment**, these two investments would produce the same return, as the investor would already know the return that he would be offered on the market in 1 year for a 1-year bond. As he also knows the current 1-year rate, he can determine the return

on a 2-year zero-coupon bond.

$$(1 + {}_0r_2)^2 = (1 + {}_0r_1) \times (1 + {}_1r_1)$$

where $_{0}r_{2}$ is the current two-year rate, $_{1}t_{1}$ the one-year rate in one year and $_{0}r_{1}$ the current 1-year rate.

The formula can be generalised for all 1-year rates:

$$(1 + {}_0r_N)^N = \prod_{t=0}^{N-1} (1 + {}_ir_1)$$

The long-term rate (*N* years) in a risk-free environment is thus the geometric average of *n* 1-year interest rates.

2/ RELATION BETWEEN INTEREST RATES AND MATURITIES IN A RISK-FREE ENVIRONMENT AND WITH NO RISK-AVERSION

If it cannot be assumed that the investor knows the 1-year rate in advance, but only that he anticipates this rate and is adopting a neutral attitude to risk, the result still produces the same type of equation.

The investor does not know with certainty the short-term rate that he will be offered in 1 year. He will therefore choose between the two options above, depending on his expectations about rates.

For example, if he believes that the short-term rate will fall sharply, he will choose the first option in order to receive a high return for 2 years.

If, on the contrary, he believes that the short-term rate will rise, he will choose the second option, in order to receive a higher return beginning the second year.

Thanks to arbitrage, the investor's hopes are the same in both options. However, this does not mean that the investor's gain will be the same after 2 years, regardless of the option chosen. Otherwise, all investors would choose the investment offering the higher return. This would raise its price and thus lower its yield, until equality between the two investments was re-established.

So, at equilibrium, this is expressed as follows:

$$(1 + {}_0r_2)^2 = (1 + {}_0r_1) \times (1 + E({}_1r_1))$$

where $_{0}t_{2}$ is the current 2-year rate and $E(_{1}t_{1})$ is the 1-year rate that is currently expected in 1 year's time.

Hence:

$$(1 + {}_0r_2) = \sqrt{(1 + {}_0r_1) \times (1 + E({}_1r_1))}$$

The 2-year rate is thus the geometric average of the current 1-year rate and current expectation of the 1-year rate a year from now.

If the current 1-year rate is 3% and the 2-year rate is 4%, the market is anticipating a 5% 1-year rate in 1 year. This is expressed as follows:

$$1.04 = \sqrt{1.03 \times (1 + E_{(1} r_{1}))}$$

i.e. $E_{(1}t_{1}) = \frac{1.04^{2}}{1.03} - 1 \approx 5\%$

Therefore, the market anticipates a rise in short-term rates. This implicit rate is called the forward rate. This formula is applied across any number of *n* periods:

The long-term rate is a **geometrical average of short-term rates** anticipated for future periods.

Mathematically, this is expressed as follows:

$$(1 + {}_{0}r_{N}) = \sqrt{(1 + {}_{0}r_{1}) \times (1 + E({}_{1}r_{1})) \times (1 + E({}_{2}r_{1})) \times \dots \times (1 + E({}_{N-1}r_{1}))}$$

where $_0r_n$ is the current rate at N year(s) and $E(_xr_1)$, is the currently expected 1-year rate in x year(s).

The shape of the yield curve provides valuable information. For example, if longterm yields are higher than short-term ones, investors are anticipating a hike in short-term interest rates.

Based on this theory, the investor must be risk-neutral. In other words, he has no preference (unlike the investors in Section 24.1) between a long-term investment or repeated short-term investments. As we saw in Section 24.1, this can happen because it is not easy to say whether the long-term or short-renewed investment strategy is riskier.

3/ INITIAL THEORIES OF RISK

The first theories to highlight the existence of a premium to reflect the relative lack of liquidity of long-term investments were the market **preferred habitat theory** and the **liquidity preference theory**.

In the mid-1960s, Modigliani and Sutch advanced the theory of preferred habitat, which says that investors prefer certain investment timeframes. Companies that wish to issue securities whose timeframe is considered undesirable, will thus have to pay a premium to attract investors.

The theory of liquidity preference is based on the same assumption, but goes further in assuming that the preferred habitat of all investors is the short term. Investors preferring liquidity will require a liquidity premium if they are to invest for the long term. Hence, long-term rates will necessarily be higher than the geometric average of anticipations of short-term rates. Even if investors anticipate fixed short-term rates, the yield curve will slope upward.

Section 24.4 THE STOCHASTIC APPROACH TO MODELLING THE RATE STRUCTURE

As the number of variables is chosen arbitrarily, the major task is to determine the variables and the random processes that they follow. The first models developed

A more sophisticated mathematical theory consists in saying that bond prices depend on a certain number of factors called state variables, which are subject to a diffusion process.

under this methodology assumed that there was only one variable of state, the short-term interest rate.

Two approaches are possible. The first consists in presenting the price of zerocoupon bonds as the solution to a differential equation. The second approach, often called "probabilistic", is based on an explicit calculation of conditional expected return under a risk-neutral probability. These methods assume that it is possible to adjust risk by adjusting probability. In this new universe, based on this different probability, all prices discounted at the risk-free rate are winners and it is enough to discount cash flow at the risk-free rate (i.e. at the very short-term rate). The second method can be used to define some requirements that the risk premium must meet, whereas in the first model, it could be chosen arbitrarily.

> Section 24.5 A FLASHBACK

After having studied the yield curve, it is easier to understand that the discounting of all the cash flows from a fixed-income security at a single rate, regardless of the period when they are paid, is an over-simplification, although this is the method that will be used throughout this text for stocks and capital expenditure. It would be wrong to use it for fixed-income securities.

In order to be more rigorous, it is necessary to discount each flow with the interest rate of the yield curve corresponding to its maturity: the 1-year rate for next year's income stream, the 3-year rate for flows paid in 3 years, etc. Ultimately, yield to maturity is similar to an average of these different rates.

The most prevalent risks associated with an investment in a debt security include the risk of default, the coupon reinvestment risk, and the risk of inflation. Relying on financial analysis, the risk of default can be isolated and analysed separately. However, the other two risks lie at opposite ends of the risk scale. Investors factor them into the risk equation through a liquidity premium, which depends on the maturity of the debt security.

Rates of return on bonds with different maturity dates can be plotted on a graph known as the yield curve. In order to avoid distortions linked to coupon rates of bonds, it is better to analyse zero coupon curves that can be reconstituted on the basis of the yield curve.

The shape of the yield curve depends on changes in expectations about short-term rates and the liquidity premium that investors will require for making a long-term investment. In a risk-free environment, the long-term rate at *n* years is a geometrical average of short-term rates anticipated for future periods. Generally, there is a positive link between the interest rate of a financial asset and its duration, which is where the rising yield curves come from. However, the yield curve can also slope the other way, especially during a recession.

Different mathematical models are now seeking to model and anticipate the shape of yield curves and how they will change on the basis of simple parameters.

SUMMARY

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QUESTIONS

@ quiz 1/What is the difference between the zero coupon curve and the yield curve?

2/Why is a yield curve showing higher long-term interest rates than short-term rates (rising curve) called a normal curve?

3/What risk are we talking about when we say that government bonds are risk-free?

4/What is the "reinvestment risk"?

EXERCISES

1/ You observe the following prices for different bonds (note that the coupons on all of them have just been paid)

Bond	Maturity (years)	Annual coupon (%)	Price
1	1	7	99
2	2	9	99
3	3	8	95
4	4	7	89
5	5	10	97

- (a) Calculate the return on each of these bonds.
- (b) Reconstruct the zero coupon curve at 5 years.
- 2/ In 2009 you notice that the yields to maturity of the same company's two bonds, 5% of 2013 and 9% of 2013, are selling at 87.44 and 100.71, respectively, and the resulting yield to maturity is 8.87% and 8.78%. You wonder why two bonds issued by the same company with the same maturity do not offer the same yield to maturity. What could be a reasonable answer?

ANSWERS

Questions

- 1/The yield curve is drawn directly, taking into account the maturity but without adjusting the coupon of each bond. The zero coupon curve is recalculated and can be used directly for valuing a security.
- 2/The preference for liquidity means that in normal circumstances (i.e. when anticipated changes in the inflation rate do not interfere), long-term rates are higher than shortterm rates.
- 3/There is no economic risk of the issuer going bankrupt.
- 4/The risk of reinvesting coupons and changes in the rate of inflation (risk of losing purchasing power).

Exercises

- 1/(a) 8.08%; 9.57%; 10.01%; 10.51%; 10.81%.
- (b) 8.08%; 9.64%; 10.09%; 10.62%; 11.01%.
- 2/The answer needs an estimation of the spot rates for the remaining years and the present value of the payments to be received. That is to say, each year's coupon and the principal payments must be discounted using the appropriate spot rates, not the yield to maturity. By doing so, we would discover that the two bonds are valued correctly by the market. Thus, the yield to maturity is different because the two bonds have different time profile of cash flow patterns: the 5% coupon bond has more cash flows coming later; the opposite occurs for the 9% bond. Yield to maturity is only a measure of convenience frequently used for sake of simplicity not the one that actually gets into the discounted cash flow of the bonds.

Period	Spot rate r _t	Coupon	PV at r _t	Coupon	PV at r _t
2010	0.06	50	47.17	90	84.91
2011	0.07	50	43.67	90	78.61
2012	0.08	50	39.69	90	71.44
2013	0.09	1050	743.85	1090	772.18
			€ 874.38		€ 1007.14

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